An Inequality Measure for Stochastic Allocations *

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Abstract

Few papers in the literature on inequality measurement deal with uncertainty. None, that we know of, provide explicit guidance on how to account for the possibility that a cohort's rank may not be fixed (e.g., socioeconomic mobility). We present a set of axioms implying such a class of inequality measures under uncertainty that is a one-parameter extension of the Generalized Gini-mean. In particular, our measure can simultaneously accommodate a preference for "shared destiny", a preference for probabilistic mixtures over unfair allocations, and a preference for fairness "for sure" over fairness in expectation.

Keywords: Other regarding, preference, social welfare, utility theory, risk, uncertainty.

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1 Introduction

Many have attributed the genesis of the modern literature on income inequality measurement to the works of Kolm (1969) and Atkinson (1970). For a given social welfare function, they define a representative level of income $r(x)$ which if distributed equally would give rise to the same level of social well being as the given income distribution x. Under the principle of transfer (Pigou, 1912; Dalton, 1920), the representative income of the income distribution x would always be less than its average income $\bar{x}$ except when societal incomes are distributed equally. This led them to define an inequality measure as $1 - r(x)/\overline{x}$. This class of inequality measures includes the Gini index, arguably the most widely used measure of income inequality, in which the representative income is given by taking the average of a decreasingly arranged distribution of incomes $\{x_1, \cdots, x_N\}$ with $2i - 1$ weight being assigned to the i^{th} richest person (Sen, 1973). The Gini representative income is then given by

$$\{x_1 + 3x_2 + \cdots + (2N - 1)x_N\}/N^2.$$

Using an additive social welfare function based on a power function, Atkinson (1970) derived a one-parameter family of inequality indices. In the same year, Rothschild and Stiglitz (1970) offered a definition of increasing risk among probability distribution functions. It is noteworthy that their definition of increasing risk mirrors the Pigou-Dalton principle of transfer which underpins much of the inequality measurement literature.

There is increasing recognition of the limitations of earlier inequality measures which, among other things, do not generally incorporate uncertainty. This is illustrated by the following example involving two individuals ($i = 1, 2$) and two equally likely states ($s = 1, 2$). An allocation in state s to individual i can be represented via the 2×2 matrix C_{si}. We seek social preferences over allocation matrices that can exhibit the following properties: For any $u, v, x, y, z \in \mathbb{R}_+$,

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix} \sim \begin{pmatrix} y & x \\ v & u \end{pmatrix} \sim \begin{pmatrix} u & v \\ x & y \end{pmatrix} \tag{1}$$

$$\underset{A}{\begin{pmatrix} \frac{z}{2} & \frac{z}{2} \\ \frac{z}{2} & \frac{z}{2} \end{pmatrix}} \succcurlyeq \underset{B}{\begin{pmatrix} z & z \\ 0 & 0 \end{pmatrix}} \succ \underset{C}{\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}} \succ \underset{D}{\begin{pmatrix} z & 0 \\ z & 0 \end{pmatrix}} \tag{2}$$

The first ranking implies indifference to the permutation of identities and a notion of *state independence* (i.e., indifference to the permutation of state labels given that states are assumed to be equally likely). The second set of rankings correspond to a *weak aversion to aggregate risk* (i.e., $A \succcurlyeq B$), followed by *a preference for shared destiny* (i.e., $B \succ C$), further followed by *a preference for ex ante fairness* (i.e., $C \succ D$). One can view these preferences as being concerned with the type of example given by Diamond (1967), where a mother wishes to allocate a good between her two children, and is restricted to an *average* allocation of $\frac{z}{2}$ per child. The mother would most prefer to give each child $\frac{z}{2}$ for sure. If this cannot be achieved, then to avoid envy and the potential for conflict amongst the children, she would prefer that each child receives the same amount in each state (hence, $B \succ C$). The least desirable allocation is the one in which one child is maximally favored for sure. Alternatively, one can view the rankings in (2) as corresponding to two unborn population cohorts who will be endowed with opportunities by the preceding generation. Allocation D corresponds to a situation where cohorts are predestined for their socioeconomic status (as, say, in a rigid class system). Allocation C corresponds to a situation where opportunities are equal for all new generations, but chance alone will ensure that the cohorts will fare unequally. Allocation B corresponds to a situation where opportunities are equal state-by-state, and Allocation A is one where opportunities are equal across individuals and states. The ranking $A \succ C \succ D$ reflects an attitude that fairness for sure dominates fairness in expectation, which in turn dominates unfairness for sure. The ranking $B \succ C$ reflects an aversion for negative correlations between individuals' outcomes (i.e., a preference for shared destiny), and the ranking $A \succcurlyeq B$ reflects weak risk aversion over aggregate outcomes. A strict preference between C and D in (2) has been modeled in the literature by Epstein and Segal (1992). Grant (1995) provides an alternative approach in the context of a decision maker who may not satisfy monotonicity. Neither approach deals with the possibility of having a preference for B over C. Ben-Porath, Gilboa, and Schmeidler (1997) offered the first formal treatment, to our knowledge, of a social preference which can exhibit such a preference for shared destiny.

Any representation that is consistent with the rankings in (1) and (2) must aggregate over both states and individuals. One natural approach to this problem is to first aggregate over states and then over individuals. Another is to first aggregate over individuals within each state and then aggregate over states. As an example, one can calculate the mean (or expected utility) over outcomes for each individual and then a Gini mean over the individual expected utilities. Alternatively, one can first calculate a Gini mean in each state, and then average over states. It is simple to see that either approach cannot exhibit the rankings in (1) and (2). Given the assumptions of symmetry, aggregating over states first would yield the same thing for allocations B and C in (2). This is generally true of a standard utilitarian approach in which information about the correlation between individuals' allocations is lost in the individual-by-individual aggregation over states. On the other hand aggregating over individuals in each state first would yield indifference between allocations C and D in (2). Thus, a "two-stage" approach to assessing fairness under uncertainty cannot satisfy (2). This observation, pointed out by Ben-Porath, Gilboa, and Schmeidler (1997), highlights the difficulty in accommodating a preference for shared destiny together with a preference for ex ante fairness.

In this paper, we present a set of axioms leading to a social welfare function that nests and extends the utilitarian two-stage approach outlined above and incorporates various considerations of social equity when allocations are stochastic. In particular, the social welfare function explicitly incorporates correlations between the individual's share and that of others, and it is through this channel that a preference for shared destiny enters. Consider again the allocation $\left(\begin{smallmatrix} x & y \\ u & v \end{smallmatrix}\right)$. Before providing an example of how our model ranks such allocations, we define some useful variables. In what follows we refer to the allocated variable as *income*, although it may equally correspond to any measurable quantity distributed among a set of individuals. Let $\tilde{w}$ denote the random variable corresponding to per capita aggregate income and $\bar{w}$ denote its mean. Define $\tilde{w}_1$ to be the column with the higher mean in $\left(\begin{smallmatrix} x & y \\ u & v \end{smallmatrix}\right)$ and $\tilde{w}_2$ be the other column (if the means are the same, then the distinction is arbitrary). Let $\tilde{x}_i = \tilde{w}_i/\tilde{w}$ be the random variable corresponding to the proportion of *per capita* aggregate income allotted to individual i (so that $\tilde{x}_i/2$ is the proportion of *total* aggregate income allocated to individual i). If $\tilde{w} = 0$ in a particular state s, then we employ the convention that $\tilde{x}_i = 1$ for each individual in state s. Denote by $\bar{x}_i$ the mean of $\tilde{x}_i$. Let $E[\cdot]$ denote an expectation over the states and $\text{Cov}(\cdot,\cdot)$ denote covariance. Restricted to matrices of the form considered above, our social welfare function V is given by:

$$V\left(\begin{smallmatrix} x & y \\ u & v \end{smallmatrix}\right) = \sum_{i=1}^{2} \left(\gamma_i - \varphi\frac{\bar{x}_i}{2}\right) E[\tilde{w}_i] + \varphi \sum_{i=1}^{2} \text{Cov}\left(\frac{\tilde{x}_i}{2}, \tilde{w} - \tilde{w}_i\right) \tag{3}$$

where $\gamma_1 < \gamma_2$, $\gamma_1 + \gamma_2 = 1$, and $\varphi \in [0, 1)$.

We note that the representation scales with its arguments. I.e., $V\left(\begin{smallmatrix} \lambda x & \lambda y \\ \lambda u & \lambda v \end{smallmatrix}\right) = \lambda V\left(\begin{smallmatrix} x & y \\ u & v \end{smallmatrix}\right)$. This makes it easy to separate considerations of absolute level of income from those of relative income inequality. When $\varphi = 0$, the representation coincides with the Generalized Gini Mean of average allocations. Thus Eq. (3) is a one-parameter extension of the two-stage aggregation procedure discussed earlier. The covariance term incorporates sensitivity to correlations – favoring random allocations for which, on average, the share of income given to an individual (i.e., $\tilde{x}_i/2$) comoves positively with a measure of the income attained by others (i.e., $\tilde{w} - \tilde{w}_i$, which is aggregated per-capita income less individual i's income).

Given that the average share and average income in allocations B and C of (2) are equal, it should be clear that it is only through the covariance term that they can be strictly ranked. For $\varphi > 0$, strict preference is given to allocations where the share of income for the poorer individual correlates positively with the rich-poor gap. In allocation B, the shares are constant and equal across individuals, so that the covariance term vanishes. By contrast, in allocation C, it is readily seen that $\tilde{w} = z/2$ is constant, and $\tilde{x}_1$ is negatively correlated with $-\tilde{w}_1 = -\tilde{x}_1 z/2$, giving rise to the strict preference for shared destiny. Specifically, for the allocations in (2) one calculates that:

$$V\left(\begin{smallmatrix} \frac{z}{2} & \frac{z}{2} \\ \frac{z}{2} & \frac{z}{2} \end{smallmatrix}\right) = \Big(1-\varphi\Big)\frac{z}{2} = V\left(\begin{smallmatrix} z & z \\ 0 & 0 \end{smallmatrix}\right) > V\left(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right) = \Big(1-2\varphi\Big)\frac{z}{2}$$
$$> V\left(\begin{smallmatrix} z & 0 \\ z & 0 \end{smallmatrix}\right) = \Big(2\gamma_1 - 2\varphi\Big)\frac{z}{2}.$$

To establish the rankings in (2), it is sufficient that $\gamma_1 < \frac{1}{2}$ and $\varphi > 0$.

The representation in (3) generalizes to an arbitrary number of individuals or cohorts in a straight forward manner. For instance, for an allocation to N equally-sized cohorts, our social welfare function V is given by:

$$V\big(\tilde{w}, \{\tilde{x}_i\}_{i=1}^N\big) = \sum_{i=1}^{N} \big(\gamma_i - \varphi\frac{\bar{x}_i}{N}\big)E[\tilde{w}_i] + \varphi\sum_{i=1}^{N} \mathrm{Cov}\Big(\frac{\tilde{x}_i}{N}, \tilde{w} - \tilde{w}_i\Big) \tag{4}$$

where the cohorts are arranged by rank so that $\bar{w}_i \geq \bar{w}_{i+1}$, and where the coefficients are restricted so that $\gamma_i < \gamma_{i+1}$, $\sum_i \gamma_i = 1$, and $\varphi \in [0, 1)$. The preference for shared destiny enters, as before, through the covariance term. To our knowledge, this is the first instance of a representation in the literature that explicitly provides guidance on how the Generalized Gini Mean can be extended to incorporate both a preference for mixing unfair allocations and a preference for shared destiny. The corresponding expression for the representative income r has a simple form: $V/(1-\varphi)$, which gives rise to a computationally tractable one-parameter extension of the generalized Gini inequality index:

$$1 - \frac{V}{(1-\varphi)\bar{w}},$$

where $\bar{w}$ is the mean per-capita allocation.

Our model is formally developed in Section 2. We adopt the usual assumptions imposed on a fairness-based preference relation (completeness, transitivity, continuity, symmetry, state independence, and a version of the Pigou-Dalton transfer principle). In addition, we require a weak preference for the convex combination of allocations and homotheticity in income. Our least normatively motivated requirements correspond to: (i) a mixture-independence style axiom for allocations that are comonotonic with respect to average-income, and (ii) a strong mixture symmetry requirement for constant aggregate income allocations that are comonotonic with respect to average-income. Finally, while our axioms completely pin down a representation for the allocations contemplated in (2) and for all allocations with constant aggregate income, we need a further requirement to extend Eq. (4) to general allocations with stochastic income. This is accomplished by requiring a duality between probabilistic mixtures and convex mixtures of random allocations. We conclude Section 2 by providing necessary and sufficient conditions for one allocation to dominate another with respect to all measures taking the form in Eq. (4). In Section 3, we discuss further properties of our inequality measure, relate it to the existing literature on income inequality measurement that incorporates uncertain allocations, and discuss how it may arise from a model of other regarding preferences in the context of decision making under uncertainty. Most proofs are provided in the Appendix.

2 Preliminaries

The set of individuals is taken to be identifiable with the closed unit interval. A Borel measurable subset of the population, $C \subseteq [0,1]$, is called a *cohort* and the sigma algebra of cohorts is denoted by $\mathcal{C}$. Sizes of cohorts are measured using the Lebesgue measure, which we denote as $m(\cdot)$. The set of payoff states is a continuous probability measure space, (Θ, Σ, μ), such that μ is convex-valued on a sigma-algebra of events, Σ.[1] Payoffs are elements of $\mathbb{R}_+$. We sometimes refer to payoffs as "income", although they can correspond to any cross-sectional attribute for which one wishes to calculate an inequality index. The choice primitives of the model are *allocation densities* — assignments of income in various events to various cohorts.[2] To that end, we consider the product space $[0,1] \times \Theta$ with its associated product sigma algebra and the product measure.

The set of allocations is the set of measurable simple functions of the form $f : [0,1] \times \Theta \mapsto \mathbb{R}_+$, and is denoted by $\mathcal{F}$.[3] One can identify the allocation $f \in \mathcal{F}$ with $\{(C_1, \tilde{w}_1), \ldots, (C_n, \tilde{w}_n)\}$ where $\{C_1, \ldots, C_n\}$ partitions the population set $[0,1]$, and $\tilde{w}_i$ is a $\mathbb{R}_+$-valued finite-ranged random variable on (Θ, Σ, μ). The interpretation is that f allocates the random income density $\tilde{w}_i$ to cohort C_i for each of $i \in 1, \ldots, n$. For any $\theta \in \Theta$, define $w(f, \theta) \equiv \int_0^1 f(p, \theta) dm(p)$ to be the aggregate *per capita* income allocated to the population in state θ.[4] Further, for $w(f, \theta) > 0$, define $x(f, p, \theta) \equiv f(p, \theta)/w(f, \theta)$ to be the *share* density of aggregate income per unit mass at p in state θ (i.e., $\int_0^1 x(f, p, \theta) dm(p) = 1$). If $w(f, \theta) = 0$ then, we will use the convention that

[1] μ is said to be convex-valued on a sigma algebra whenever for any $A \in \Sigma$ and $\alpha \in (0,1)$ there is some $a \in \Sigma$ such that $a \subset A$ and $\mu(a) = \alpha\mu(A)$.

[2] The notion of a *density* is consistent with spreading an aggregate amount of finite income over a continuum of "individuals" in $[0,1]$.

[3] A simple function from $[0,1] \times \Theta$ to $\mathbb{R}_+$ takes the form $\sum_{i=1}^K \mathbf{1}_{D_i} v_i$, where K is finite, $\mathbf{1}_{D_i}$ is an indicator function whose value is 1 on the measureable set $D \subseteq [0,1] \times \Theta$ and zero elsewhere, and $v_i \in \mathbb{R}_+$. Restricting attention to simple functions allows for a straight-forward method of generating a probabilistic mixtures of allocations, and also ensures that the utility of any admissible allocation is finite. As it turns out, our final representation is finite for any allocation in the intersection of L^1 and L^2 but this is a result of our axioms which we do not wish to anticipate in the definition of the primitives of choice.

[4] Because the population is normalized to have a unit measure, it makes sense to think of $w(f, \theta)$ as per capita.

$x(f, p, \theta) = 1$ for all $p \in [0, 1]$. We abuse notation by identifying $w > 0$ with the act $f(p, \theta) = w$ for every $(p, \theta) \in [0, 1] \times \Theta$.

For any $f, g \in \mathcal{F}$ and $\lambda, \nu \in \mathbb{R}_+$, the allocation density $\lambda f + \nu g \in \mathcal{F}$ is defined by $\big(\lambda f + \nu g\big)(p, \theta) = \lambda f(p, \theta) + \nu g(p, \theta)$ for each $(p, \theta) \in [0, 1] \times \Theta$. Unless stated otherwise, the topology of $[0, 1] \times \Theta$ is assumed to be the topology of weak convergence.[5]

Intuitively, the allocations f and g are random variables whose payoffs are allocations across the population. A probabilistic or "α-mixture" of f and g is also a random allocation that is much like a two-stage lottery: In the first stage a coin is tossed whose probablity of "heads" is $\alpha \in [0, 1]$. In the second stage, if the coin is "heads" then the population allocation will be determined by the random allocation f, while if the coin is "tails" it will be determined by the random allocation g. The continuous nature of Θ and the fact that μ is convex-valued enables the construction of α-mixtures from any two allocations, $f, g \in \mathcal{F}$. This can be done as follows. For $f, g \in \mathcal{F}$, let $\mathcal{P}(f, g)$ be the coarsest partition of $[0, 1] \times \Theta$ that is adapted to both f^{-1} and g^{-1}. Each element in $\mathcal{P}(f, g)$ is assigned a unique combination of payoffs by f and g. Because μ is convex-valued, for each $\alpha \in (0, 1)$, any $P \in \mathcal{P}(f, g)$ has a subset, $P_\alpha \in \mathcal{C} \times \Sigma$, such that $\alpha\mu(P) = \mu(P_\alpha)$. Construct a new partition, $\hat{\mathcal{P}} \equiv \hat{\mathcal{P}}_\alpha \cup \hat{\mathcal{P}}_{1-\alpha}$ where $\hat{\mathcal{P}}_\alpha \equiv \{P_\alpha \mid P \in \mathcal{P}(f, g)\}$ and $\hat{\mathcal{P}}_{1-\alpha} \equiv \{P \setminus P_\alpha \mid P \in \mathcal{P}(f, g)\}$. Now assign each $P_\alpha \in \hat{\mathcal{P}}_\alpha$ the outcome $f(P)$ and each $P_{1-\alpha} \in \hat{\mathcal{P}}_{1-\alpha}$ the outcome $g(P)$. As a random variable, the resulting allocation is equivalent to the two-stage lottery described earlier and is what we formally mean when referring to an α-mixture of f and g.[6]

Example 1. Consider the cohort $C = [0, \frac{1}{2}]$ and its complement. Consider, also, an event in Θ, say $E \in \Sigma$, that has mass $\mu(E) = \frac{1}{2}$. Let f be the allocation represented by the following matrix:

$$f = \begin{array}{c} \\ E \\ E^c \end{array} \begin{array}{c} \begin{array}{cc} C & C^c \end{array} \\ \left(\begin{array}{cc} y & 2w - y \\ z & 2w - z \end{array} \right) \end{array}.$$

y is the density of payoffs allocated to cohort C in event E. The total income allotted to C in event E is $\frac{y}{2}$. Note that the sum of the payoffs in each event, weighted by cohort size, is w which is the aggregate per capita income. The allocation f can also be identified with $\{(C, \tilde{w}_1), (C^c, \tilde{w}_2)\}$, where the $\tilde{w}_i$'s are correlated 50:50 binomial random variables, such that $\tilde{w}_1$ awards y when $\tilde{w}_2$ pays $2w - y$, and z when $\tilde{w}_2$ pays $2w - z$. Correspondingly, the share densities (i.e., the $\tilde{x}_i$'s) are correlated 50:50 binomial random variables, such that $\tilde{x}_1$ awards y/w when $\tilde{w}_2$

[5] If the topologies of weak convergence on $[0, 1]$ and (Θ, Σ, μ) are denoted by $W^*([0, 1])$ and $W^*(\Theta)$, respectively, then the topology of weak convergence on $[0, 1] \times \Theta$ is the product topology of $W^*([0, 1])$ and $W^*(\Theta)$.

[6] It should be clear that there are many ways to construct an α-mixture from two allocations, $f, g \in \mathcal{F}$. This indeterminacy is immaterial as our axioms require state-independence, meaning that only payoffs and their associated probabilities matter for ranking allocations.

pays $2 - y/w$, and z/w when $\tilde{w}_2$ pays $2 - z/w$. The sum of the share densities, weighted by cohort size, is 1 in each state.

Suppose g allocates an income density of $2w$ to cohort $D = [\frac{1}{4}, \frac{3}{4}]$ in every state, and 0 to its complement. Then for any $\lambda \in [0,1]$,

$$\lambda f + (1-\lambda) g = \begin{array}{c} \\ E \\ E^c \end{array} \begin{array}{cccc} C \cap D & C \cap D^c & C^c \cap D & C^c \cap D^c \\ \end{array} \left(\begin{array}{cccc} \lambda y + 2(1-\lambda) w & \lambda y & \lambda(2w - y) + 2(1-\lambda) w & \lambda(2w-y) \\ \lambda z + 2(1-\lambda) w & \lambda z & \lambda(2w - z) + 2(1-\lambda) w & \lambda(2w-z) \end{array} \right).$$

Now, suppose that $e, e' \in \Sigma$ with $e \subset E$, $e' \subset E^c$, and $\mu(e) = \mu(e') = \frac{\alpha}{2}$. Define,

$$\begin{array}{cccc} C \cap D & C \cap D^c & C^c \cap D & C^c \cap D^c \end{array} \quad (5)$$

$$f_\alpha g = \begin{array}{r} e \\ e' \\ E \setminus e \\ E^c \setminus e' \end{array} \begin{pmatrix} y & y & 2w - y & 2w - y \\ z & z & 2w - z & 2w - z \\ 2w & 0 & 2w & 0 \\ 2w & 0 & 2w & 0 \end{pmatrix}. \quad (6)$$

Then $f_\alpha g$ is an α-mixture of f and g. Intuitively, the population has a probability of α of being allocated f and a probability of $1 - \alpha$ being allocated g. ∎

Let $E_\mu[\cdot]$ denote an unconditional expectation operator with respect to the measure μ. For any $f \in \mathcal{F}$, define $\bar{f} \equiv E_\mu[f]$ to be the allocation density that pays to each cohort of f its expected payoffs. In Example 1, $\bar{f}$ pays $\frac{y+z}{2}$ per unit mass to cohort C in every state of the world, and pays $2w - \frac{y+z}{2}$ per unit mass to cohort C^c in every state of the world. Because $\bar{f}$ is constant across states, we suppress its θ-dependence (i.e., we write $\bar{f}(p)$ instead of $\bar{f}(p, \theta)$).

For any $f, g \in \mathcal{F}$, define

$$\Delta(f) \equiv \Big\{ g \in \mathcal{F} \mid (\bar{f}(p) - \bar{f}(p'))(\bar{g}(p) - \bar{g}(p')) \geq 0 \ \forall p, p' \in [0,1] \Big\}.$$

In the parlance of the related literature in choice theory, $g \in \Delta(f)$ if the mean payoff of g to each of its cohorts is *comonotonic* with the mean payoff of f to each of its cohorts. In other words, f and g are mean-comonotonic if and only if they agree on the ranking of cohorts according to their mean payoffs. We term $\Delta(f)$ the *mean-comonotonic cone* of f.

Finally, we define various proper subsets of $\mathcal{F}$. Let

$$\mathcal{F}_{NS} = \{ f \in \mathcal{F} \mid f(p, \theta) = \bar{f}(p) \ \forall \theta \in \Theta \text{ and } p \in [0,1] \}$$

be the set of non-stochastic allocations in $\mathcal{F}$. Further, let

$$\mathcal{F}_{NSW} = \{f \in \mathcal{F} \mid w(f,\theta) = w(f,\theta') \, \forall \theta, \theta' \in \Theta\}$$

be the set of allocations with non-stochastic aggregate income.

3 Axioms

Consider a social planner's preferences over $\mathcal{F}$, corresponding to the binary relation $\succcurlyeq$ and satisfying the following basic conditions.

B1 (Ordering)**.** *$\succcurlyeq$ is complete, transitive, continuous, and $w \succ w'$ whenever $w > w'$.*[7]

Completeness, transitivity, and continuity are standard assumptions on non-strict preference relations such as $\succcurlyeq$. The monotonicity required by Condition B1 is likewise a normative assertion that a large fair allocation is always preferred to a smaller fair allocation.

B2 (Null allocations)**.** *If f and g in $\mathcal{F}$ differ only on a set of $\mu \times m$-measure zero, then $f \sim g$.*

Condition B2 asserts that only allocations to non-negligible cohorts in non-negligible events matter.

B3 (Symmetry and state independence)**.** *Suppose $f \in \mathcal{F}$ can be identified with $\{(C_1, \tilde{w}_1), \ldots, (C_n, \tilde{w}_n)\}$ and $g \in \mathcal{F}$ can be identified with $\{(C'_1, \tilde{w}'_1), \ldots, (C'_n, \tilde{w}'_n)\}$, and that all the C_i's and C'_i's have positive m-measure. Suppose further that there exists a permutation of $1, \ldots, n$, denoted by $\pi(\cdot)$, such that $\forall i \; m(C_{\pi(i)}) = m(C'_i)$ and the joint distribution function of $(\tilde{w}_{\pi(i)}, \ldots, \tilde{w}_{\pi(n)})$ is equal to the joint distribution function of $(\tilde{w}'_i, \ldots, \tilde{w}'_n)$. Then $f \sim g$.*

Condition B3 requires that the only relevant attribute of a cohort is its size, and the only relevant attribute of an event is its probability. In particular, this means that if $f, g, h, h' \in \mathcal{F}$ such that h and h' are both α-mixtures of f and g (corresponding to the same α), then $h \sim h'$ — i.e., there is indifference between all equivalent probabilistic mixtures. Moreover, Axiom B3 requires indifference to permuting the identity of two equally sized cohorts.

Conditions, B1-B3 are "basic" to a ranking of social allocations over anonymous individuals. We now introduce a set of axioms over $\succcurlyeq$ that further pin down the preference structure.

[7] By "continuous" we mean that the sets $\{g \mid g \succcurlyeq f\}$ and $\{g \mid f \succcurlyeq g\}$ are closed in the topology of weak convergence $\forall f, g \in \mathcal{F}$.

Axiom 1 (Transfer Principle)**.** *Let $f \in \mathcal{F}$ be a non-stochastic allocation having non-null cohorts $\{C_1, \ldots, C_n\}$, indexed such that $f(C_1) > \ldots > f(C_n)$. Suppose g is a non-stochastic allocation obtained from f by transferring a positive amount of income from cohort i to cohort $i+1$, while maintaining $g(C_1) \geq \ldots \geq g(C_n)$ and $w(f, \theta) = w(g, \theta)\ \forall \theta \in \Theta$. Then $g \succ f$.*

Axiom 1 asserts a version of the Pigou-Dalton transfer principle which is central to the income inequality literature.

Axiom 2 (Scale Invariance)**.** *For any $\lambda \in \mathbb{R}_+$ and $f, f' \in \mathcal{F}$, if $f \succcurlyeq f'$ then $\lambda f \succcurlyeq \lambda f'$.*

Axiom 2 is needed for analyzing relative inequality across populations with different levels of aggregate incomes. In other words, it allows one to "normalize" allocations by a scale factor, say the mean per-capita allocation, so that relative inequality can be compared. Such an assumption is particularly useful in modeling steady-state inequality distributions in the presence of growth and is an attribute of the class of Generalized Gini Means.

Axiom 3 (Mean-comonotonic Independence)**.** *For any $f \in \mathcal{F}$, and $f', g \in \Delta(f)$,*

$$f \succcurlyeq f' \;\Leftrightarrow\; f_\alpha g \succcurlyeq f'_\alpha g,$$

where $f_\alpha g$ and $f'_\alpha g$ are α-mixtures with $\alpha \in (0,1)$.

In words, Axiom 3 asserts the independence axiom of expected utility theory whenever the stochastic allocations involved are mutually mean-comonotonic. Variants of this axiom have appeared in other formulations of rank-dependent utility (see, for example, Weymark, 1981; Chew and Wakker, 1996; Yaari, 1987). Because the the set of fair allocations (i.e., allocations which are uniform across the population in every state) is a subset of every mean-comonotonic cone, Axiom 3 implies that $\succcurlyeq$ satisfies the independence axiom on the set of fair allocations. In particular, because an α-mixture of a fair allocation is also a fair allocation, the implication is that $\succcurlyeq$ can be represented by an expected utility functional when restricted to the set of fair allocations.[8]

Whereas Axiom 3 is intended to pin down the attitude of the social planner towards the variation of allocation across states of nature (i.e., risk), the next axiom targets the attitude towards variation across individuals. Consider the set of non-stochastic allocations to a partition of $[0,1]$ consisting of n equally sized cohorts, which we denote by $\mathcal{F}_{NS}(n)$. Consider next the principle of Weak Independence of Income Source (WIIS), introduced by Weymark (1981): For all comonotonic $f, g, h \in \mathcal{F}_{NS}(n)$,

$$f \succcurlyeq g \;\Leftrightarrow\; f + h \succcurlyeq g + h.$$

[8]Specifically, if a fair allocation is denoted by its associated per-capita random allocation $\tilde{w}$, then Axioms 2 and 3 together imply the expected utility representation $E[\frac{\tilde{w}^\gamma - 1}{\gamma}]$ over fair allocations.

Weymark (1981) essentially demonstrates that WIIS, together with Conditions B1-B3 and Axiom 1 imply the well-studied Generalized Gini mean ranking, taking the form $G(f) \equiv \sum_{i=1}^{N} \gamma_i w_i$, where $(w_1, \ldots, w_N)$ is a decreasing rank-ordering of the allocations made by $f \in \mathcal{F}_{NS}(n)$, and the γ_i's are increasing. In words, WIIS asserts that among similarly rank ordered allocations, an inequality comparison depends only on the difference between the allocations being compared.[9]

In placing restrictions on the social planner's attitudes towards allocational variation across individuals we consider two weakenings of WIIS, extended to include stochastic allocations:

Axiom 4′ (Mean-comonotonic Betweenness)**.** *Suppose $f, g \in \Delta(f)$ have the same aggregate per capita income, which is constant across states. Then for any $\alpha \in (0,1)$, $f \sim g$ implies $\alpha f + (1-\alpha)g \sim g$.*

Axiom 4′, restricted to $\mathcal{F}_{NS}$ is strictly weaker than WIIS (i.e., it implies WIIS but is not implied by WIIS) in several ways.[10] The axiom only applies to allocations with the same aggregate wealth. Moreover, the invariance to commonalities between the allocations applies only to equivalent (via indifference) allocations, and only to commonalities propotional to g or f (as opposed to arbitrary comonotonic commonalities — h in WIIS). If $\succcurlyeq$ is restricted to $\mathcal{F}_{NS}(n)$, then (together with B1-B3 and Axiom 1) Axiom 4′ does not lead to a Generalized Gini mean representation. However, as we shall shortly demonstrate, enriching the choice space to include stochastic allocations and imposing Axiom 3 recovers the Generalized Gini mean over $\mathcal{F}_{NS}(n)$.

The axioms proposed thus far comprise a "marriage" of the axiom commonly used to pin down attitudes towards risk (i.e., Axiom 3) and a weakening of the axiom commonly used to pin down attitudes towards allocational inequity (i.e., Axiom 4′). Thus, the normative features of these axioms in their respective domains can be said to carry over to the conjunction of the two domains. Despite that, and the fact that Axiom 4′ is a weakening of WIIS, it is still too strong to permit non-neutral attitudes towards shared destiny. To see this, consider the allocations depicted in Eq. (2) of the Introduction. Moreover, define $B' \equiv \left(\begin{smallmatrix} 0 & 0 \\ z & z \end{smallmatrix}\right)$ and $C' \equiv \left(\begin{smallmatrix} 0 & z \\ z & 0 \end{smallmatrix}\right)$. Then, because A, B, B', C and C' are all mean-comonotonic, it is straight forward to see that Axioms 3 and 4′ imply the ranking

$$B' \sim B \sim \frac{1}{2}B' + \frac{1}{2}B = A = \frac{1}{2}C' + \frac{1}{2}C \sim B \sim B'.$$

Thus, $A \sim B \sim C$, and there is no allowance for a strict preference for shared destiny as desired. To obtain the latter, one must further weaken 4′. To that end, we first point out that Axiom 4′ implies that $\alpha f + (1-\alpha)g \sim \alpha' g + (1-\alpha')f$ for any $\alpha, \alpha' \in (0,1)$. We now consider the following:

[9] A weaker but similar principle was invoked in Ben Porath and Gilboa (1994) to derive the Generalized Gini mean.

[10] Just as WIIS has its counterpart in the literature on individual decision making under risk (i.e., the Independence Axiom), Betweeness too has its counterpart (See Chew, 1989; Dekel, 1986).

Axiom 4 (Mean-comonotonic Strong Mixture Symmetry)**.** *Suppose $f, g \in \Delta(f)$ have the same aggregate per capita income, which is constant across states. Then for any $\alpha \in (0,1)$, $f \sim g$ implies $\alpha f + (1-\alpha)g \sim \alpha g + (1-\alpha)f$.*

Axiom 4 weakens Axiom 4′ to only assert indifference between *symmetric* shifts in the two equivalent comonotonic allocations, f and g. Notice that when applied to B and B' (or C and C'), Axiom 4 is simply reasserting the symmetry principle already implied by Axiom 3. The Axiom is silent, however, with respect to how $A = \frac{1}{2}B' + \frac{1}{2}B$ (or $\frac{1}{2}C' + \frac{1}{2}C$) are related to B or C. In other words, there is no apparent contradiction in applying Axioms 3 and 4, and the ranking $A \succcurlyeq B \succ C$.

Before moving on to the representation theorems, it is worth pointing out that Axiom 4 also has its counterpart in the theory of individual decision making under risk (see Chew, Epstein, and Segal, 1991). In particular, it was adopted in this context by Epstein and Segal (1992) to extend a Harsanyi-like utilitarian social welfare function to include a preference for the probabilistic mixture of unfair allocations.[11]

4 Representations

Definition 1. For any non-stochastic $f \in \mathcal{F}_{NS}$, the decumulative distribution function of f is $D_f(z) \equiv m\Big(\{c | f(c,\Theta) \geq z\}\Big)$. The cumulative distribution of f is $F_f(z) \equiv 1 - D_f(z)$.

Our first result employs Axiom 4′ to extend the Generalized Gini mean to situations where allocations are stochastic.

Theorem 1. *Assume $\succcurlyeq$ satisfies conditions B1-B3. Then the following are equivalent:*

i) Axioms 1-3 and 4′

ii) $\forall f, g \in \mathcal{F}_R$, $f \succcurlyeq g \;\Leftrightarrow\; V(f) \succcurlyeq V(g)$ where

$$V(f) = \int_0^\infty G\big(D_{\bar{f}}(z)\big)dz, \tag{7}$$

and $G : [0,1] \mapsto [0,1]$ is some continuous and strictly convex function such that $G(1) = 1$ and $G(0) = 0$.

[11] Because of the different context in which Epstein and Segal (1992) use mixture symmetry, their representation does not accommodate a strict preference for shared destiny.

All proofs appear in the Appendix.

$V(f)$ is essentially the well-known Generalized Gini Mean (Weymark, 1981), but applied to the mean of cohort income (recall that $\bar{f}$ corresponds to the mean allocation of f). One can interpret Axioms 1-3 and Axiom $4'$ as providing a foundation for such a class of inequality measures. In the special case where f consists of N equally-sized cohorts such that f assigns the random variable $\tilde{w}_i$ with mean $\bar{w}_i \equiv E_\mu[\tilde{w}_i]$ to the ith cohort, Theorem 1 reduces to

$$V(f) = \sum_{i=1}^{N} \eta_i \bar{w}_i.$$

The cohorts are indexed so that $\bar{w}_1 \geq \bar{w}_2 \ldots \geq \bar{w}_N$, while the η_i's sum to one and are strictly increasing. The fact that η_i is increasing with i implies inequality aversion because cohorts with higher average income densities contribute less to the sum. An important instance of this measure is the "Gini-mean over means" where $\eta_i = \frac{\hat{\eta}_i}{\sum_{j=1}^{N} \hat{\eta}_j}$ and $\hat{\eta}_i = 2i - 1$.

It is important to emphasize that a rank-dependent measure over means does not trivialize uncertainty. To see this, consider the allocations D and D' both in $\mathcal{F}_R$ and discussed earlier. It should be clear that for any $\alpha \in (0, 1)$, the α-mixture of D with D' dominates both D and D', and that the most desirable mixture has $\alpha = \frac{1}{2}$. This is satisfying because it addresses the criticism of Diamond (1967), allowing probabilistic mixing to serve as a fairness inducing mechanism. The problem, however, is that the representation in Theorem 1 neglects two important facets of uncertainty and its intuitive influence on fairness. Firstly, Eq. (7) implies indifference between the half-half mixture of D and D' (i.e., allocation C), and the allocation A that guarantees equality in every state. Although there may be instances where this can be descriptive, it seems natural to allow for aversion to anticipated inequality and not only aversion to average inequality. Secondly, Eq. (7) says nothing about how correlations between individuals might matter. One might wish, for instance, to further discount an allocation that awards the least fortunate (in expectations) cohort an allocation that is *negatively* correlated with the remaining cohorts.

Preference for shared destiny

We take the view that Axioms 1-3 and $4'$ provide a useful benchmark for assessing an inequality adjusted welfare measure under uncertainty, and one that bridges the settings with and without uncertainty. While these axioms, and therefore the representation, appear to be natural in introducing uncertainty into inequality measurement considerations, a more satisfactory treatment of aversion to anticipated inequality and non-neutrality towards inter-cohort correlations calls for a further weakening of the axioms. As argued in the last section, symmetry

(Condition B3) and comonotonic Betweeness (Axiom 4$'$) preclude non-neutral attitudes towards shared destiny, and the most arbitrary of these two Axioms is arguably the latter. We weaken the Mean-comonotonic Betweenness axiom to Mean-comonotonic Strong Mixture Symmetry, leading to the following and central result of the paper.

Theorem 2. *Assume $\succcurlyeq$ satisfies conditions B1-B3. Then the following are equivalent:*

i) Axioms 1-4

ii) $\forall f, g \in \mathcal{F},\ f \succcurlyeq g \ \Leftrightarrow\ V(f) \succcurlyeq V(g)$ *where either*

$$V(f) = \nu \int_{\Theta} w(f,\theta)^{\zeta}\, d\mu(\theta) - \varphi \int_{[0,1]} \int_{\Theta} w(f,\theta)^{\zeta} x(f,p,\theta)^2\, d\mu(\theta)\, dm(p) \tag{8}$$

with $\varphi > 0$ *and* $(\nu - \varphi)\zeta > 0$,

or

$$V(f) = \int_{0}^{\infty} G\big(D_{\bar{f}}(w)\big) dw - \varphi \int_{[0,1]} \int_{\Theta} f(p,\theta) x(f,p,\theta)\, d\mu(\theta)\, dm(p) \tag{9}$$

where $G : [0,1] \mapsto [0,1]$ *is some continuous and convex function such that* $G(0) = 0$ *and* $G(1) = 1$, *and* $0 \le \varphi < 1$. *Moreover, if* $\varphi = 0$ *then* $G(\cdot)$ *is strictly convex.*

Note that the two representations agree if $\zeta = 1$ in Eq. (8) and $G(x) = x$ in Eq. (9). Moreover, the two admissible representations in (8) and (9), it is easy to demonstrate that only the latter is compatible with a strict preference for ex-ante fairness (e.g., $C \succ D$, where C and D are the allocations in (2) of the Introduction). Hence, the following Corollary, which insists on a strict preference for both ex-ante and ex-post:

Corollary to Theorem 2. *Assume $\succcurlyeq$ satisfies conditions B1-B3. Then the following are equivalent:*

i) Axioms 1-4, and $\succcurlyeq$ exhibits a strict preference for both ex-ante and ex-post fairness

ii) $\forall f, g \in \mathcal{F},\ f \succcurlyeq g \ \Leftrightarrow\ V(f) \succcurlyeq V(g)$ *where* $V(f)$ *is given by Eqn.* (9), $G : [0,1] \mapsto [0,1]$ *is some continuous and strictly convex function such that* $G(0) = 0$ *and* $G(1) = 1$, *and* $0 < \varphi < 1$.

If the population consists of N equal-sized cohorts, then the representation reduces to

$$V(f) = \sum_{i=1}^{N} \gamma_i \bar{w}_i - \varphi \sum_{i=1}^{N} E[\frac{\tilde{x}_i}{N} \tilde{w}_i], \tag{10}$$

where the cohorts are indexed so that $\bar{w}_i \geq \bar{w}_{i+1}$, while the γ_i's are strictly increasing in i and sum to one. The expression in Eq. (4) follows by noting that $\sum_i \frac{\tilde{x}_i}{N} = 1$. Thus, in weakening Axiom 4′ to 4, one obtains a *one-parameter* extension of the Generalized Gini representation over means. As shown in the Introductory examples, this is sufficient to account for a ranking in which perfect equality is preferred to equality in expectations, which in turn is preferred to static inequality (e.g., $A \succ C \succ D$, in the Introduction). Moreover, the representation admits a preference for shared destiny (e.g., $B \succ C$, also in the Introduction).

Sketch of proof

In the proof, we first restrict attention to allocations for which the aggregate income is constant. Axiom 3 allows one to further restrict attention to the mean rank-ordered cone. Intuitively, one can think of each allocation as a matrix (e.g., Eq. 5). Application of Axiom 4 (resp. Axiom 4′) then implies that the representation is a quadratic (resp. linear) aggregator of the matrix elements. The Independence-style Axiom 3 and "state independence" (Condition B3) then imply that the matrix elements are aggregated across events only in proportion to the probability of the events. Continuity (Condition B1) and Symmetry (Condition B3) are then used to reduce the quadratic term to its diagonal elements, which must all share the same coefficient. Next, Axiom 1 is shown to require that more weight be given to the linear components of poorer cohorts (in expectation) and that the quadratic term be non-positive.

Next we, employ Axiom 2 to show that when income is constant (in the mean rank-ordered cone), the representation is homothetic in income. When income is random, application of the Independence Axiom, along with Continuity and Symmetry, implies that the homotheticity is of degree 1, and that the representation applies to all elements of the mean rank-ordered cone.

4.1 Dominance Criteria

Theorem 2 offers a possible way of ranking two allocations consistent with the Diamond (1967) principle and the principle of shared destiny. However, using a single measure to rank to allocations may not be satisfactory if one is seeking a robust way of ranking allocations. In the empirical literature on measuring income inequality, the primary approach to robustly ranking allocations is through the criterion of second degree stochastic dominance. This is applied to a *non-stochastic* distribution of income as follows:

Definition 2. For any two non-stochastic allocations with equal aggregate income, $f, g \in \mathcal{F}_{NS}$, f is said to Second Degree Stochastically Dominate (SSD) g if and only if for every $w \in \mathbb{R}_+$,

$$\int_0^w D_f(w')dw' \geq \int_0^w D_g(w')dw'. \tag{11}$$

One can likewise ask whether there is a robust way of ranking *stochastic* allocations for the class of measures implied by Theorem 2. Specifically, suppose $f, g \in \mathcal{F}$. Under what conditions will $f \succcurlyeq g$ for every $\succcurlyeq$ satisfying Axioms 1-4? The answer is particularly simple:

Proposition 1. *The following are equivalent:*

i) $f \succcurlyeq g$ *for every ranking satisfying Conditions B1-B3 and Axioms 1-4.*

ii) The distribution of means implied by f *second-degree stochastically dominates the distribution of means implied by* g*, and*

$$\int_{[0,1]} \int_{\Theta} f(p,\theta)x(f,p,\theta)\, d\mu(\theta)\, dm(p) \geq \int_{[0,1]} \int_{\Theta} g(p,\theta)x(g,p,\theta)\, d\mu(\theta)\, dm(p). \tag{12}$$

The intuition for Proposition 1 is as follows. One first establishes that $f \succcurlyeq g$ for every ranking based on the Generalized Gini-over-means whenever $\bar{f}$ second degree stochasticaly dominates $\bar{g}$. Then, because $\succcurlyeq$ is a one-parameter extension of a Generalized Gini-over-means, and the extension corresponds to adding the φ term in Eq. (9), the dominance criterion must also include this term independently (as in Eq. (12)). Thus, to extend the standard SSD comparison of distributions to our settings, one need only check the additional dominance criterion specified in Eq. (12).

4.2 Other regarding preference

Besides inequality measurement, the social welfare function axiomatized here can be applied to model individual decision making under risk. For instance, consider an individual, subscripted by p, with attitudes towards stochastic allocations described by

$$V_p(f) = a_p \int_{\Theta} f(p,\theta)d\mu(\theta) \; + \int_0^{\infty} G_p\big(D_{\bar{f}}(w)\big)dw - \varphi_p \int_{[0,1]} \int_{\Theta} f(p',\theta)x(f,p',\theta)\, d\mu(\theta)\, dm(p'), \tag{13}$$

with $a_p > 0$ for all $p \in [0, 1]$. Note that the representation in Eq. (13) satisfies a modified form of conditions B1-B3 and Axioms 1-4 where symmetry and the transfer principle apply to all other individuals (but not to individual p). The other-regarding utility function in Eq. (13) captures an individual's self interest (through the first term) as well as concerns for inequality along the lines pursued thus far. Aggregating over all individual utilities (i.e., integrating over $p \in [0.1]$), one obtains a utilitarian welfare function satisfying the requirements of Theorem 2.[12] Thus, one can view the welfare function derived in Theorem 2 as the result of a utilitarian approach to welfare when individuals possess other-regarding preferences.

5 Inequality Measure

In the decision theory literature, the "certainty equivalent" of a random payoff distribution is some sure amount, such that the decision maker is indifferent between receiving the random payoff and the sure amount. In the social choice literature, pioneered by Kolm (1969) and Atkinson (1970), "states" are reinterpreted as "individuals". I.e., instead of referring to a payoff distribution across states, one refers to an income distribution across individuals. Likewise, the Atkinson-Kolm definition of an "equally distributed equivalent representative income" corresponds to a "certainty equivalent" in the decision theory literature.

In extending the model to uncertain allocations, we have axiomatized a social welfare function which incorporates a sensitivity towards correlation between the individual's share of income and the relative income of others. For constant aggregate income, should the mixture symmetry axiom be strengthened to a "betweenness"-style requirement (as in Theorem 1), the resulting representation essentially reduces to the generalized Gini mean (Weymark, 1981; Yaari, 1987) defined over distributions of mean incomes.

Assuming $\succcurlyeq$ is represented by function $V(\cdot)$, of the form given in Eq. (10), an equally distributed sure allocation of w per capita has utility of $(1 - \varphi)w$. Thus it is sensible to define the representative income $r(f)$ for our social welfare function V by:

$$r(f) \equiv V(f)/(1 - \varphi), \quad \varphi \in [0, 1), \tag{14}$$

containing the special case of a generalized Gini mean over means when $\varphi = 0$. This in turn gives rise to the class of inequality measures:

$$I_V(f) \equiv 1 - r(f)/\bar{f}, \tag{15}$$

[12] Define $G(x) \equiv \frac{1}{\int_{[0,1]}(a_p+1)dp} \int_{[0,1]} \Big(a_p x + G_p(x)\Big) dp$, and $\varphi \equiv \frac{1}{\int_{[0,1]}(a_p+1)dp} \int_{[0,1]} \varphi_p dp$.

where $\bar{f}$ denotes the mean income across individuals as well as states. The class of inequality measures, $I_V(\cdot)$, exhibits the standard properties of being homogenous of degree 0 and vanishing when there is complete equality. Moreover, $-I_V$ is ordinally equivalent to V among allocations with the same aggregate wealth. For $\varphi = 0$, (15) reduces to a generalized Gini income inequality index over distributions of mean incomes.[13] In that special case, only the distribution of mean income matters and, from standard results on the Generalized Gini Mean, $I_V(f) = 0$ if and only if the distribution of mean income across individuals is constant almost everywhere. For $\varphi > 0$, the following Proposition demonstrates that fairness for sure always dominates fairness in expectation, i.e., the most equitable allocation is fair in virtually every state of nature.

Proposition 2. *Assume that $\varphi > 0$. Then $I_V(f) = 0$ if and only if $x(f, p, \theta) = 1$ for almost every $(p, \theta) \in [0, 1] \times \Theta$.*

The proposition further clarifies that a non-neutral preference for shared destiny arises from the φ term, which on a state-by-state basis favors an allocation of equal shares. At the same time, the φ term does not distinguish between fixed and randomized identities when a sure but unfair allocation is assigned (e.g., D versus C from the Introduction). However, the generalized Gini-over-means portion of $V(\cdot)$ does differentiate between these cases and favors randomizing over identities (which narrows the distribution of expected outcomes). Thus, whereas the Generalized Gini portion of the representation favor fairness à la Diamond (1967), the φ portion favors shared destiny and introduces a non-neutral attitude towards inter-personal correlations.

It is noteworthy that when $\varphi > 0$, I_V can be unbounded from above as societal income gets concentrated in one infinitesimal individual. An allocation for which $I_V(f) > 1$, will have a *negative* representative income. Such an allocation induces social envy to the point where reducing aggregate income, and thus the disparity between rich and poor, may be preferable to the status quo. Indeed, experiments such as those conducted by Charness and Rabin (2002) provide empirical confirmation for the presence of such social attitudes.[14]

5.1 Calibration

Consider allocations among two equally sized cohorts of the form discussed in the Introduction. As discussed in the Introduction, for any positive $z \neq y$, non-indifference between the allocation

[13] When $\varphi = 0$ and $G(p) = p^2$, the representation further reduces to the standard Gini mean defined over distributions of mean incomes.

[14] One can rule out the presence of envy if the allocation space is restricted to a finite number of individuals (or cohorts) and φ is sufficiently small. In doing so, the upper bound on φ will depend on the number individuals as well as the γ_i coefficients.

$\left(\begin{smallmatrix} z & z \\ y & y \end{smallmatrix}\right)$ and the allocation $\left(\begin{smallmatrix} z & y \\ y & z \end{smallmatrix}\right)$ corresponds to non-neutral attitudes concerning shared destiny. Assuming a concern for shared destiny, one can measure the magnitude of such a preference by finding a quantity $\varepsilon \geq 0$ that render the following two allocations equally palatable:

$$\begin{pmatrix} z-\varepsilon & z-\varepsilon \\ y-\varepsilon & y-\varepsilon \end{pmatrix} \sim \begin{pmatrix} z & y \\ y & z \end{pmatrix}. \tag{16}$$

The representative income of the allocation on the left is $r = (z+y)/2 - \varepsilon$. Thus, the quantity ε can be viewed as a compensating representative income that arises because of a preference for shared destiny. Equating the utility of these two allocations using the representation in Eq. (9) results in

$$\varepsilon = \frac{1}{2}\frac{\varphi}{1-\varphi}\frac{(z-y)^2}{z+y}, \tag{17}$$

confirming that the parameter φ is a measure of affinity for shared destiny. When φ is close to one, ex-post envy is severe and the compensating representative income is large.

One can also invert this relationship and calibrate φ from observed other-regarding preferences to obtain,

$$\varphi = \frac{\varepsilon}{\varepsilon + \frac{(z-y)^2}{z+y}}. \tag{18}$$

An experiment designed to elicit compensating representative income (i.e., ε in Eq. (16)) can be used to calibrate φ to prevailing social sentiments concerning shared destiny. In addition, the elicited φ will be independent of the functional form of $G(\cdot)$ in Theorem 2. In particular, it may be natural to employ the prevalent inequality measure, the Gini Mean corresponding to the case of $G(p) = p^2$, and supplement it with the calibrated φ term.

6 Concluding remarks

We bring together and formally extend axioms from two distinct literatures: choice theory for individuals facing risky prospects, and the theory of measuring allocational inequality in a non-stochastic setting. By doing so, we derive an inequality measure for stochastic allocations that simultaneously accommodates non-neutral attitudes to both ex-ante *and* ex-post fairness.

Our model is related to several strands of social choice and decision theory literatures. The original Harsanyi (1955) approach of aggregating individual utilities into a single welfare function faces the difficulty that it neglects both the benefits of randomization over individuals when an allocation is unfair (the Diamond, 1967, critique), and the possibility that inter-personal correlations can matter (e.g., shared destiny). Epstein and Segal (1992) first employed the strong mixture symmetry axiom to derive a social welfare function which can exhibit a preference for ex ante fairness, thereby addressing the Diamond (1967) critique.[15] Their formulation, however, is still in the Harsanyi tradition of aggregating over individual utilities and thus ignores inter-personal correlations by construction.

Other papers in the literature consider the notion of ex-post fairness, which in this paper we term "shared destiny." The difficulty of obtaining a simultaneous preference for ex-ante and ex-post fairness has been documented in Ben-Porath, Gilboa, and Schmeidler (1997) and Gajdos and Maurin (2004). The latter reference establishes that reasonable assumptions on welfare orderings rule out two-stage aggregation over states and cohorts. Fleurbaey (2007), recognizing this difficulty, argues in favor of ex-post preferences. Additional discussion of this is presented in two of the papers in this volume (see Grant, Kajii, Polak, and Safra, 2010; Fleurbaey, Gajdos, and Zuber, 2010).[16]

To our knowledge, the first paper to theoretically consider inter-personal correlations is Ben-Porath, Gilboa, and Schmeidler (1997), who introduce uncertain allocations into their social welfare function using multiple priors. Their representation, in the case of allocations of the form $\left(\begin{smallmatrix} u & v \\ x & y \end{smallmatrix}\right)$ takes the form,

$$V\left(\begin{smallmatrix} u & v \\ x & y \end{smallmatrix}\right) = \min_{p \in P} \Big(p_1 u + p_2 v + p_3 x + p_4 y \Big), \tag{19}$$

where $p = (p_1, \ldots, p_4)$ is a probability distribution and P is a closed and convex set of probability distributions. With little structure on the set P which encodes the behavioral properties of V, there is no ready interpretation of the representation or how one should restrict attention to a subclass in order to exhibit some specific ranking. Though not implied by their axioms, the following example illustrates how the social welfare function in (19) can exhibit a preference for shared destiny as well as a preference for ex ante fairness (i.e., the $B \succ C \succ D$ ranking in Eq. (2)

[15] Grant (1995) provides an alternative approach in the context of a single decision maker who may not exhibit monotonicity.

[16] There are many treatments in the social choice literature that do not specifically focus on correlations or shared destiny but consider social welfare under uncertainty. Recent works include Gajdos (2002), Gajdos and Maurin(2004), Gajdos, Tallon, and Verhnaud (2008). Gajdos and Tallon (2002) consider specifically ex-post versus ex-ante "envy" in the measurement of social welfare, although their focus is more on the ex-ante efficiency allocations when avoidance of ex-post envy is a desideratum. Gajdo, Weymark and Zoli (2009) study a particular (binary) structure for uncertainty in order to introduce a sensitivity to shared destiny when assessing fatality in social risk. The literature on multi-dimensional Generalized Gini indices is also related to this (Gajdos and Weymark, 2005; Decancq, 2009), in that each state of nature can be viewed as a particular dimension along which one might wish to assess a Generalized Gini mean (the indices in Gajdos and Weymark, 2005, are based on two-stage aggregation and thus do not capture the shared destiny effect.

of the Introduction). Let

$$\hat{P} = \Big\{(\frac{5}{24}, \frac{7}{24}, \frac{1}{24}, \frac{11}{24}), (\frac{1}{24}, \frac{11}{24}, \frac{5}{24}, \frac{7}{24}), (\frac{7}{24}, \frac{5}{24}, \frac{11}{24}, \frac{1}{24}), (\frac{11}{24}, \frac{1}{24}, \frac{7}{24}, \frac{5}{24})\Big\},$$

and set P to be the closed convex hull of $\hat{P}$. Under these assumptions,

$$V(A) = V(B) = \frac{z}{2} > V(C) = \frac{z}{3} > V(D) = \frac{z}{4}.$$

While P gives the desired ranking, it is hard to interpret in a manner that can readily be generalized to an arbitrary number of cohorts and states.

Another paper that offers a model admitting both ex-post and ex-ante preferences over social allocations is Saito (2010) who, in the classic Anscomb-Aumann approach, explicitly differentiates between ex-ante and ex-post preferences in his axioms. The resulting elegant representation is a convex mixture over ex-post and ex-ante maximin preferences.

A Appendix

Proof of Theorem 1: See the proof of Theorem 2 and, in particular, the treatment of the linear region. ∎

Proof of Theorem 2: To prove that the Axioms are equivalent to the utility representation in (9), consider first the restriction of $\mathcal{F}$ to random variables, adapted to an even number, greater than two, of equally likely events in Θ (with respect to μ) and allocated to $N > 1$ equally sized cohorts in $[0, 1]$ such that the aggregate per capita income allocated in each state is a constant, $w > 0$. I.e., each element of the restriction of $\mathcal{F}$ is an N-vector of random variables, $\{\tilde{x}_i\}_{i=1}^N$ defined over $|S|$ equally likely states, such that $\frac{1}{N}\sum_{i=1}^N \tilde{x}_i = w$. Denote the space of associated N-vector random variables as $\mathbf{X}(w, N, S)$ (Condition B3 ensures that there is no loss of generality in not specifying the partitions of Θ and $[0, 1]$ into S equally-likely events and N equally-sized cohorts, respectively).

Refer to each event in the partition of Θ as a state. We abuse notation by referring to the index set of states as S. The number of states is denoted as $|S|$. For each $\mathbf{x} \in \mathbf{X}(w, N, S)$, label the payoff to the ith individual in state $s \in S$ as x_{is}. In the following, we focus only on random vectors in the mean-ordered cone, Δ_0, for which $\pi(\mathbf{x})$ is the identity (Condition 3 implies that there is no loss of generality in doing so). We proceed by proving several results, under the assumption that Conditions B1-B3, as well as Axioms 3 and 4 hold.

Proposition A.1. *When restricted to $\Delta_0 \cap \mathbf{X}(w, N, S)$, $\succcurlyeq$ has a representation of the form,*

$$V(\mathbf{x}) = \begin{cases} \sum_{i>1} \gamma_i \Big(\frac{1}{|S|}\sum_s x_{is}\Big) + \sum_{ij>1} \phi_{ij} \Big(\frac{1}{|S|}\sum_s x_{is}x_{js}\Big) \\ \qquad\qquad + \sum_{ij>1} \hat{\phi}_{ij} \Big(\frac{1}{|S|}\sum_s x_{is}\Big)\Big(\frac{1}{|S|}\sum_{s'} x_{js'}\Big) + \rho & \text{if } \mathbf{x} \succ \mathbf{x}', \\ \\ \sum_{i>1} \eta_i \Big(\frac{1}{|S|}\sum_s x_{is}\Big) & \text{if } \mathbf{x}' \succcurlyeq \mathbf{x} \succcurlyeq \mathbf{x}'', \\ \\ \sum_{i>1} \gamma'_i \Big(\frac{1}{|S|}\sum_s x_{is}\Big) + \sum_{ij>1} \phi'_{ij} \Big(\frac{1}{|S|}\sum_s x_{is}x_{js}\Big) \\ \qquad\qquad + \sum_{ij>1} \hat{\phi}'_{ij} \Big(\frac{1}{|S|}\sum_s x_{is}\Big)\Big(\frac{1}{|S|}\sum_{s'} x_{js'}\Big) + \rho & \text{otherwise.} \end{cases} \tag{20}$$

for some $\mathbf{x}', \mathbf{x}'' \in \Delta_0 \cap \mathbf{X}(w, N, S)$.

Proof. Each element of $\mathbf{X}(w, N, S)$ can be thought of as a vector in $\mathbb{R}^{N|S|}$. In light of the discussion in Chew, Epstein, and Segal (1991) (see, especially, Appendices 2 & 3), Condition B1 and Axiom 4 imply that $\Delta_0 \cap \mathbf{X}(w,N,S)$ is partitioned into three regions, a "middle" one in which indifference surfaces are linear (solving the equation $c = V(\mathbf{x})$ where $V(\cdot)$ is linear), separating two regions in which indifference surfaces quadratic — the more preferred region being strictly convex and the least preferred region being strictly concave (solving the equation $c = V(\mathbf{x})$ where $V(\cdot)$ is quadratic).[17] Denote the linear region of $\Delta_0 \cap \mathbf{X}(w,N,S)$ as $\mathbf{X}_{LR}$ and assume, without loss of generality, that $\mathbf{x} \in \mathbf{X}_{LR} \Rightarrow \mathbf{x}' \succcurlyeq \mathbf{x} \succcurlyeq \mathbf{x}''$. Denote any one of the quadratic regions as $\mathbf{X}_{QR}$.

The Linear Region: Each indifference surface in $\mathbf{X}_{LR}$ can be parameterized as $c_x = \frac{1}{|S|}\sum_{i>1}\sum_s x_{is}\eta_{is}(c_x)$, with c_x constant along an indifference surface.[18] Consider now an arbitrary allocation, $\mathbf{x}$, in the interior of $\mathbf{X}_{LR}$, and for which $c_x = \frac{1}{|S|}\sum_{i>1}\sum_s x_{is}\eta_{is}(c_x)$ for some c_x.[19] Fix some $s, s' \in S$ and $j > 1$ and assume that not all the $\eta_{is''}(c_x)$'s for $s'' \neq s, s'$ are zero (recall that $|S| > 2$ by assumption). For some $\epsilon > 0$, find $\hat{\mathbf{x}} \in \mathbf{X}_{LR}$ that can be written as $\hat{x}_{is''} = x_{is''} + \delta_{is''}$, for every $i > 1$ and $s'' \neq s, s'$, such that $\frac{1}{|S|}\sum_{i,s''\neq s,s'}\delta_{is''}\eta_{is''}(c_x) = -\frac{\epsilon}{|S|}\eta_{js}(c_x)$; $\hat{x}_{js} = x_{js} + \epsilon$; and otherwise $\hat{x}_{is''} = x_{is''}$. One can always find such an ϵ and associated $\delta_{is''}$'s because $\mathbf{x}$ is interior. A quick calculation establishes that $\hat{\mathbf{x}} \in \mathbf{X}_{LR}$ and is on the indifference surface of $\mathbf{x}$. However, by Condition B3 so must be the allocations generated from $\mathbf{x}$ and $\hat{\mathbf{x}}$ by swapping states s and s'. Swapping states s and s' in $\mathbf{x}$ therefore yields

$$c_x = \frac{1}{|S|}\sum_{i>1}\sum_{s''\neq s,s'} x_{is''}\eta_{is''}(c_x) + \frac{1}{|S|}\sum_{i>1}\big(x_{is'}\eta_{is}(c_x) + x_{is}\eta_{is'}(c_x)\big),$$

while swapping states s and s' in $\hat{\mathbf{x}}$ yields

$$c_x = \frac{1}{|S|}\sum_{i>1}\sum_{s''\neq s,s'} x_{is''}\eta_{is''}(c_x) - \frac{\epsilon}{|S|}\eta_{js}(c_x) + \frac{1}{|S|}\sum_{i>1}\big(x_{is'}\eta_{is}(c_x) + x_{is}\eta_{is'}(c_x)\big) + \frac{\epsilon}{|S|}\eta_{js'}(c_x).$$

Subtracting the two equations, yields $\eta_{js'}(c_x) = \eta_{js}(c_x)$ whenever not all the $\eta_{is''}(c_x)$'s (for $s'' \neq s, s'$) are zero.

If all the $\eta_{is''}(c_x)$'s (for $s'' \neq s, s'$) are zero, then it cannot be that $\eta_{js}(c_x)$ and $\eta_{js'}(c_x)$ are both non-zero (one just picks one of the s'''s instead of s', run the argument used earlier, and this will force the contradiction $\eta_{js}(c_x) = 0$). So, if all the $\eta_{is''}(c_x)$'s (for $s'' \neq s, s'$) are zero *and* $\eta_{js'}(c_x) = 0$, then consider the allocation $\hat{\mathbf{x}}$ that adds $\epsilon > 0$ to $x_{js'}$ such that $\bar{x}_j + \frac{\epsilon}{|S|} < \bar{x}_{j-1}$, so that $\hat{\mathbf{x}} \in \mathbf{X}_{LR}$. Because $\eta_{js'}(c_x) = 0$, $\hat{\mathbf{x}}$ is on the indifference surface of $\mathbf{x}$. Swapping the states s and s' for both $\mathbf{x}$ and $\hat{\mathbf{x}}$, and using Condition B3, leads to (through an argument similar to the

[17] The arguments in Chew, Epstein, and Segal (1991) pertain to the unit simplex in $\mathbb{R}^N$. By comparison, $\mathbf{X}(w,N,S)$ is an n-product of simplices, because $\frac{1}{N}\sum_i x_{is} = w$ for every $s \in S$, and $x_{is} \in \mathbb{R}_+ \ \forall i, s$. However, their results readily extend to any closed and convex subset of a Euclidean space with a non-empty interior, such as $\Delta_0 \cap \mathbf{X}(w,N,S)$.

[18] The summing-up constraint in each state makes it unnecessary to include the allocation of the wealthiest individual.

[19] For $\mathbf{x}$ to be in the interior of $\mathbf{X}_{LR}$ means that $x_{is} > 0$ for every $i > 1$ and $s \in S$, and that $\bar{x}_1 > \ldots > \bar{x}_N$.

one made earlier) $\epsilon\eta_{js}(c_x) = 0$, which is a contradiction. Summarizing, if all the $\eta_{is''}(c_x)$'s (for $s'' \neq s, s'$) are zero then $\eta_{js'}(c_x) = \eta_{js}(c_x) = 0$. Thus in all cases, and all $j > 1, s, s' \in S$, $\eta_{js'}(c_x) = \eta_{js}(c_x) \equiv \eta_j(c_x)$.

One can now parameterize the indifference surface without loss of generality as $c_x = \sum_{i>1} \bar{x}_i \eta_i(c_x)$, where $\bar{x}_i \equiv \frac{1}{|S|} \sum_s x_{is}$ is the average payoff of cohort i. Conditions B1 (continuity) and Axiom 1 then imply that the $\eta_i(c_x)$'s are increasing. One can take c_x to be a utility measure with the representation

$$\mathbf{x} \succcurlyeq \mathbf{y} \;\Leftrightarrow\; c_x \geq c_y.$$

Consider now the indifference surface through $\mathbf{x}$, a non-zero non-stochastic allocation in the interior of $\mathbf{X}_{LR}$, having utility of c. The fact that S has an even number of equally likely states ensures that the set

$$\mathcal{E}(c, \mathbf{x}) \equiv \left\{ \mathbf{y} \in \mathbf{X}_{LR} \mid \mathbf{y}_{\frac{1}{2}}\mathbf{x} \sim \mathbf{x} \right\}$$

is not empty. In fact, $\mathcal{E}(c, \mathbf{x})$ is at least $N - 2$ dimensional because it contains non-stochastic allocations of the form $\mathbf{x} + \boldsymbol{\varepsilon}$ where $0 = \sum_{i>1} \bar{\varepsilon}_i \eta_i(c)$.[20] Now consider a non-stochastic allocation $\mathbf{x}'$ constructed from $\mathbf{x}$ by distributing $\epsilon \neq 0$ from the wealthiest cohort to another cohort. By Condition B1, $\mathbf{x}$ and $\mathbf{x}'$ are strictly ordered and therefore do not lie on the same indifference surface. Moreover, $c = \sum_{i>1} \bar{x}_i \eta_i(c) \neq \sum_{i>1} \left(\frac{1}{2}\bar{x}_i + \frac{1}{2}\bar{x}'_i \right) \eta_i(c)$, so that $\mathbf{x}_{\frac{1}{2}}\mathbf{x}'$ and $\mathbf{x}$ are also strictly ordered. Let c' be the utility measure of $\mathbf{x}_{\frac{1}{2}}\mathbf{x}'$. Because Axiom 3 implies that $\mathbf{y}_{\frac{1}{2}}\mathbf{x}' \sim \mathbf{x}_{\frac{1}{2}}\mathbf{x}'$ for every $\mathbf{y} = \mathbf{x} + \boldsymbol{\varepsilon} \in \mathcal{E}(c, \mathbf{x})$, it must be that

$$0 = \sum_{i>1} \bar{\varepsilon}_i \eta_i(c) \;\Leftrightarrow\; 0 = \sum_{i>1} \bar{\varepsilon}_i \eta_i(c').$$

In turn, this can only be true if the $N - 1$ vector of $\eta_i(c)$'s is proportional to the $N - 1$ vector of $\eta_i(c')$'s. This establishes that, within a neighborhood of $\mathbf{x}$, all indifference surfaces are parallel. It should be clear that this argument can be extended to all non-stochastic allocations in the interior of $\mathbf{X}_{LR}$ by "patching" together open neighborhoods around any interior non-stochastic allocation. Because the representation in $\mathbf{X}_L$ depends only on the means of individual allocations (i.e., all indifference surfaces correspond to some non-stochastic allocation), the argument holds for a dense set in $\{c_x \mid \mathbf{x} \in \mathbf{X}_{LR}\}$. Continuity then implies that the $\eta_i(\cdot)$'s are colinear on all of $\mathbf{X}_{LR}$. In particular, the representation in $\mathbf{X}_{LR}$ can be written, after normalization, as

$$c_x = \sum_{i>1} \eta_i \bar{x}_i. \tag{21}$$

[20] Because $\mathbf{x}$ is interior, one can always find a neighborhood of zero in $\mathbb{R}^{N-1}$ so that the vector of $\bar{\varepsilon}_i$'s lies in this neighborhood, $0 = \sum_{i>1} \bar{\varepsilon}_i \eta_i(c)$, *and* the non-stochastic allocation endowing cohort i with $x_i + \bar{\varepsilon}_i$ is in the interior of $\mathbf{X}_{LR}$.

The Quadratic Region: Chew, Epstein, and Segal (1991) prove that each indifference surface in $\mathbf{X}_{QR}$ can be parameterized as $c_x = \sum_{ii'ss'} \phi_{ii'ss'} x_{is} x_{i's'} + \sum_{is} \gamma_{is} x_{is} + \rho$, where i and j run from 2 to N and $\phi_{ii'ss'} = \phi_{i'is's}$. Consider an allocation, $\mathbf{x} \in \mathbf{X}_{QR}$, that endows each individual $i > 1$ with an allocation of $0 < x < w$. Pick an arbitrary i, s, and s', and consider the allocation $\mathbf{z} \in \mathbf{X}_{QR}$ generated from $\mathbf{x}$ by shifting $0 < \epsilon < x$ from individual 1 to individual i in state s, and $-\epsilon$ from individual 1 to individual i in state $s' \neq s$. Let $\mathbf{z}'$ be the allocation that permutes the allocation $\mathbf{z}$ in states s and s'. Condition B3 implies that $\mathbf{z} \sim \mathbf{z}'$. Subtracting the utility of $\mathbf{z}'$ from that of $\mathbf{z}$ yields

$$2\epsilon(\gamma_{is} - \gamma_{is'}) + 4\epsilon x \sum_{jr} \Big(\phi_{ijsr} - \phi_{ijs'r}\Big) = 0.$$

Because x and ϵ could be chosen to be arbitrarily small, it must be that $\gamma_{is} = \gamma_{is'}$ for all i, s and s'. This allows us to rewrite $c_x = \sum_{ii'ss'} \phi_{ii'ss'} x_{is} x_{i's'} + \sum_i \gamma_i \bar{x}_i + \rho$.

Now consider an arbitrary allocation $\mathbf{x} \succ \mathbf{x}'$ that only pays individual i the striclty decreasing amount x_i (for $i = 2, \ldots, N$) in state s and zero otherwise. The utility of this allocation is $\frac{1}{|S|} \sum_{i>1} \gamma_i x_i + \sum_{ik} \phi_{ikss} x_i x_k + \rho$. Permuting state s and some arbitrary state s', and using Condition B3 leads to

$$0 = \sum_{ik>1} x_i x_k \Big(\phi_{ikss} - \phi_{iks's'}\Big).$$

Because the x_i's can be locally varied arbitrarily whilst keeping them striclty decreasing, it must be that $\phi_{ikss} = \phi_{iks's'}$ for any $i, k > 1$ and $s, s' \in S$. Now consider an arbitrary allocation $\mathbf{y}$ that only pays individuals $i > 1$ in states s and $s' \neq s$ the amounts y_i and y_i', respectively, such that $y_i + y_i' > y_{i+1} + y_{i+1}'$. The utility of $\mathbf{y}$ is

$$\frac{1}{|S|} \sum_{i>1} \gamma_i (y_i + y_i') + \sum_{ik>1} \Big(y_i y_k \phi_{ikss} + y_i' y_k' \phi_{iks's'} + 2 y_i y_k' \phi_{ikss'}\Big) + \rho.$$

Consider a permutation of the allocation $\mathbf{y}$ that takes s to some r and s' to some $r' \neq r$. Subtracting the utilities of $\mathbf{y}$ and its permutation and using Condition B3 gives

$$0 = \sum_{ik>1} \Big(y_i y_k \big(\phi_{ikss} - \phi_{ikrr}\big) + y_i' y_k' \big(\phi_{iks's'} - \phi_{ikr'r'}\big) + 2 y_i y_k' \big(\phi_{ikss'} - \phi_{ikrr'}\big)\Big).$$

Using our earlier deduction, that $\phi_{ikss} = \phi_{iks's'}$ for any $i, k > 2$ and $s, s' \in S$, yields $0 = \sum_{ik>1} y_i y_k' \big(\phi_{ikss'} - \phi_{ikrr'}\big)$. Because $y_i y_k'$ can be varied locally arbitrarily (without violating the ranking $y_i + y_i' > y_{i+1} + y_{i+1}'$), it must be that $\phi_{ikss'} = \phi_{ikrr'}$ for every $i, k > 1$ and $s, s', r, r' \in S$ such that $s \neq s'$ and $r \neq r'$. Consequently, after some manipulation we can rewrite the representation in $\mathbf{X}_{QR}$ as

$$c_x = \sum_{i>1} \gamma_i \bar{x}_i + \sum_{ij>1} \phi_{ij} \Big(\frac{1}{|S|} \sum_s x_{is} x_{js}\Big) + \sum_{ij>1} \hat{\phi}_{ij} \bar{x}_i \bar{x}_j. \tag{22}$$

Because the arguments here apply to both quadratic regsions, this is sufficient to establish the representation in Eq. (20). ∎

Let $\mathbf{X}(w,N) \subset \mathcal{F}$ be the space of random allocations to N equally likely cohorts such that the total aggregate (per capita) income in any event is w. Let $E[\cdot]$ denote the expectation operator over Θ with respect to μ. Finally, let $\tilde{x}_i$ denote the random allocation to cohort i associated with $\mathbf{x} \in \mathbf{X}(w,N)$.

Proposition A.2. *When restricted to* $\mathbf{X}(w,N) \cap \Delta_0$, $\succcurlyeq$ *has a representation of the form,*

$$V(\mathbf{x}) = \sum_{i=1}^{N} \gamma_i E[\tilde{x}_i] - \varphi \sum_{i=1}^{N} E[\tilde{x}_i^2], \tag{23}$$

where $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_N$ *and* $\varphi \geq 0$.

Proof. The representation in Eq. (20) holds for *any* random allocation that is adapted to $|S|$ equally likely states, with $|S|$ arbitrary and aggregate wealth constant across states. The set of such random vectors is dense in $\mathbf{X}(w,N) \cap \Delta_0$. Continuity therefore implies that the representations in (21) and (22) can be extended to all of $\mathbf{X}(w,N) \cap \Delta_0$:

$$V(\mathbf{x}) = \begin{cases} \sum_{i>1} \gamma_i E[\tilde{x}_i] + \sum_{ij>1} \phi_{ij} E[\tilde{x}_i \tilde{x}_j] + \sum_{ij>1} \hat{\phi}_{ij} E[\tilde{x}_i]^2 + \rho & \text{if } \mathbf{x} \succ \mathbf{x}', \\ \sum_{i>1} \eta_i E[\tilde{x}_i] & \text{if } \mathbf{x}' \succcurlyeq \mathbf{x} \succcurlyeq \mathbf{x}'' \\ \sum_{i>1} \gamma'_i E[\tilde{x}_i] + \sum_{ij>1} \phi'_{ij} E[\tilde{x}_i \tilde{x}_j] + \sum_{ij>1} \hat{\phi}'_{ij} E[\tilde{x}_i]^2 + \rho & \text{otherwise,} \end{cases}$$

for some $\mathbf{x}' \in \mathbf{X}(w,N) \cap \Delta_0$. Axiom 3 requires that the representation be expected utility (up to a monotonic transformation) with respect to probability distributions in (Θ, Σ, μ). This requires that the utility function in the two regions coincide. Moreover, to be an expected utility functional (up to a monotonic transformation) in the quadratic region, either the $\hat{\phi}$ term must be zero or the γ and ϕ terms must be zero. In the latter case, the representation must be equivalent to a linear one; i.e., $\sum_{ij>1} \hat{\phi}_{ij} E[\tilde{x}_i]^2 = \left(\sum_{i>1} \tau_i E[\tilde{x}_i] \right)^2$. Whatever the case, there is an equivalent representation for $\succcurlyeq$ having the following general form:

$$V(\mathbf{x}) = \sum_{i>1} \gamma_i E[\tilde{x}_i] + \sum_{ij>1} \phi_{ij} E[\tilde{x}_i \tilde{x}_j]. \tag{24}$$

Let $\sigma_{i,j} \equiv \mathrm{Cov}(\tilde{x}_i, \tilde{x}_j)$ for $i \neq j$, and $\sigma_i^2 = \mathrm{var}(\tilde{x}_i)$. Suppose $N \geq 3$. Fix $i \geq 1$ and $j > 1$ such that $j < N$ and $i \neq j, j+1$, and consider an allocation $\mathbf{x}$ for which $\bar{x}_j = \bar{x}_{j+1}$, $\sigma_{i,j}, \sigma_j^2, \sigma_i^2 \neq 0$, while $\sigma_{l,k} = \sigma_l^2 = 0$, otherwise. The contribution to $V(\mathbf{x})$ from $\sigma_{i,j}$ and σ_j^2 is $\phi_{ij}\sigma_{i,j} + \phi_{jj}\sigma_j^2$ (where $\phi_{ij} = 0$ by definition if $i = 1$). Because $\bar{x}_j = \bar{x}_{j+1}$, Condition B3 requires that the utility remains unchanged when one exchanges the payoffs of individuals j and $j+1$ in every state (keeping in mind that such an exchange keeps the allocation in $\mathbf{X}(w,N)\cap\Delta_0$. For this to be true, it must be that $\phi_{ij}=\phi_{i,j+1}$ and that $\phi_{jj}=\phi_{j+1,j+1}$. Applying this to all $1<i,j<N$, implies that the representation has the form,

$$V(\mathbf{x})=\sum_{i>1}\gamma_i E[\tilde{x}_i]-\hat{\varphi}\sum_{i>1}E[\tilde{x}_i^2]-\hat{\phi}\sum_{\substack{ij>1\\ i\neq j}}E[\tilde{x}_i\tilde{x}_j]. \tag{25}$$

If $N=2$, then Eq. (24) reduces to Eq. (25), so Eq. (25) holds generally on $\Delta_0\cap\mathbf{X}(w,N)$. Because $\frac{1}{N}\sum_{i=1}^{N}\tilde{x}_i=w$, one can rewrite Eq. (25), up to a constant, as

$$V(\mathbf{x})=\sum_{i=1}^{N}\gamma_i E[\tilde{x}_i]-\varphi\sum_{i>1}E[\tilde{x}_i^2]-\phi E[\tilde{x}_1^2]. \tag{26}$$

If $N=2$, then because of the summing-up constraint, one can write the representation as in Eq. (23) (up to a constant). Assume, therefore, that $N>2$ and consider an allocation $\mathbf{x}\in\Delta_0\cap\mathbf{X}(w,N)$ that is non-stochastic for all cohorts save for some $i>1$ and $i+1$, such that each cohort receives the same mean allocation density (i.e., w), cohort i is endowed with $\tilde{x}_i=\bar{x}_i+\tilde{\varepsilon}$, where $\tilde{\varepsilon}$ has mean zero, and cohort $i+1$ is endowed with $\bar{x}-\tilde{\varepsilon}$. Exchanging the payoffs of individuals 1 and i leaves the allocation in $\Delta_0\cap\mathbf{X}(w,N)$ and, by Axiom 3, must yield the same utility as $\mathbf{x}$. This necessitates $\phi=\varphi$ and establishes the functional form in the Proposition. To derive the restrictions on the coefficients, consider a non-stochastic allocation with $x_i>x_{i+1}$ and consider a transfer from cohort i to cohort $i+1$ that doesn't change the allocational ranking. Axiom 1 implies that this is a strictly preferred transfer, meaning that in utility terms $(\gamma_{i+1}-\gamma_i)\Delta+2\varphi\big(\Delta-(x_i-x_{i+1})\big)\Delta>0$, where $\Delta>0$ is the magnitude of the transfer. Essential arbitrariness of $(x_i-x_{i+1})>$ and $\Delta>0$ require that the γ_i's be increasing and that φ be non-negative. ∎

Proposition A.2 applies to $\mathbf{X}(w,N)$ with w arbitrary. It therefore applies, specifically, to the case where $w=1$ and $\mathbf{x}\in\mathbf{X}(1,N)$ can also be viewed as an allocation of shares. Henceforth, we denote any element of $\mathbf{X}(w,N)$ as $w\mathbf{x}$ where $\mathbf{x}\in\mathbf{X}(1,N)$. Axiom 2 implies that if $\mathbf{x}\succcurlyeq\mathbf{x}'$ for $\mathbf{x},\mathbf{x}'\in\mathbf{X}(1,N)$ then $w\mathbf{x}\succcurlyeq w\mathbf{x}'$. Thus the restriction of $\succcurlyeq$ to $\Delta_0\cap\mathbf{X}(1,N)$ completely determines the restriction of $\succcurlyeq$ to $\Delta_0\cap\mathbf{X}(w,N)$. In particular, because the representation of $\succcurlyeq$ in $\Delta_0\cap\mathbf{X}(w,N)$ is expected utility with respect to (Θ,Σ,μ), it must be that the functional representation in Eq. (23) for w is equivalent to the one with $w'\neq w$, up to an affine transformation. Consequently, for any $w\mathbf{x}$ where $w>0$ and $\mathbf{x}\in\mathbf{X}(1,N)$,

$$V(w\mathbf{x})=a(w)\Big(\sum_{i=1}^{N}\gamma_i E[\tilde{x}_i]-\varphi\sum_{i=1}^{N}E[\tilde{x}_i^2]\Big)+b(w), \tag{27}$$

where continuity implies that $a(w)$ and $b(w)$ are continuous in w, and $a(w)a(w') > 0 \; \forall w, w' \in \mathbb{R}_+$.

Define $\mathbf{X}(N) \equiv \bigcup_{w>0} \mathbf{X}(w, N)$.

Proposition A.3. *When restricted to* $\mathbf{X}(N)$, $\succcurlyeq$ *has a representation of the form:*

$$V(w\mathbf{x}) = w^{\zeta}\Big(\sum_{i=1}^{N} \gamma_i E[\tilde{x}_i] - \varphi \sum_{i=1}^{N} E[\tilde{x}_i^2] + b\Big),$$

where $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_N$ and $\varphi \geq 0$.

Proof. Suppose that there exists a neighborhood $\mathcal{N} \subset \mathbb{R}_+$ in which $a(\cdot)$ or $b(\cdot)$ vary with income (if there isn't such a neighborhood, then set $\zeta = 0$ and the proof is done). Fix $w, w' \in \mathcal{N}$ and $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in \mathbf{X}(1, N)$ such that $w\mathbf{x} \sim w'\mathbf{x}'$ and $w\mathbf{y} \sim w'\mathbf{y}'$, while $w \neq w'$, $\mathbf{x} \not\sim \mathbf{y}$ (so that $\mathbf{x}' \not\sim \mathbf{y}'$), (continuity and the fact that $\succ$ is non-empty, by Axiom 1, ensures that one can do this). Let $\xi \equiv \sum_{i=1}^{N} \gamma_i E[\tilde{x}_i] - \varphi \sum_{i=1}^{N} E[\tilde{x}_i^2]$, $\xi' \equiv \sum_{i=1}^{N} \gamma_i E[\tilde{x}_i'] - \varphi \sum_{i=1}^{N} E[\tilde{x}_i'^2]$, $\eta \equiv \sum_{i=1}^{N} \gamma_i E[\tilde{y}_i] - \varphi \sum_{i=1}^{N} E[\tilde{y}_i^2]$, and $\eta' \equiv \sum_{i=1}^{N} \gamma_i E[\tilde{y}_i'] - \varphi \sum_{i=1}^{N} E[\tilde{y}_i'^2]$. One can therefore write,

$$b(w) + \xi a(w) = b(w') + \xi' a(w'), \tag{28}$$

$$b(w) + \eta a(w) = b(w') + \eta' a(w'). \tag{29}$$

Employing Axiom 2, it must be the case that

$$b(\lambda w) + \xi a(\lambda w) = b(\lambda w') + \xi' a(\lambda w'), \tag{30}$$

$$b(\lambda w) + \eta a(\lambda w) = b(\lambda w') + \eta' a(\lambda w'). \tag{31}$$

Subtracting Eq. (29) from Eq. (28), and Eq. (31) from Eq. (30) gives

$$a(w)(\xi - \eta) = a(w')(\xi' - \eta')$$
$$a(\lambda w)(\xi - \eta) = a(\lambda w')(\xi' - \eta').$$

By assumption, $\xi \neq \eta$ and $\xi' \neq \eta'$, so one can divide the former equation by the latter to conclude that $\frac{a(\lambda w)}{a(w)} = \frac{a(\lambda w')}{a(w')}$ and that $\log\big(a(w)\big)$ satisfies Cauchy's Equation. Corollary 5 to Theorem 3 in Aczel and Dhombres (1989), Chapter 2, implies that $a(w) = aw^{\zeta}$. Applying this to Eqns. (28) and (29) gives

$$b(w) - b(w') = a(\xi' w'^{\zeta} - \xi w^{\zeta})$$
$$b(\lambda w) - b(\lambda w') = a\lambda^{\zeta}(\xi' w'^{\zeta} - \xi w^{\zeta}).$$

If it so happens that $\xi' w'^{\zeta} = \xi w^{\zeta}$ then $\log b(w)$ also satisfies Cauchy's Equation, and $b(w) = bw^{\zeta}$ — the exponent must be the same as in the case of $a(w)$ otherwise Axiom 2 could not hold. If $\xi' w'^{\zeta} \neq \xi w^{\zeta}$ then

$$\frac{b(\lambda w) - b(\lambda w')}{b(w) - b(w')} = \lambda^{\zeta}. \tag{32}$$

In this case, set $\hat{b}(w) \equiv b(w) - b(1)$, and by setting $w' = 1$ rewrite Eq. (32) as

$$\hat{b}(\lambda w) = \lambda^{\zeta} \hat{b}(w) + \hat{b}(\lambda). \tag{33}$$

Fix any $\bar{w} > 1$ and write,

$$\begin{aligned}
\hat{b}(x) =& \Big(\frac{x}{\bar{w}}\Big)^{\zeta} \hat{b}(\bar{w}) + \hat{b}\big(\frac{x}{\bar{w}}\big) \\
=& \Big(\frac{x}{\bar{w}}\Big)^{\zeta} \hat{b}(\bar{w}) + \Big(\frac{x}{\bar{w}^2}\Big)^{\zeta} \hat{b}(\bar{w}) + \hat{b}\big(\frac{x}{\bar{w}^2}\big) \\
=& \Big(\frac{x}{\bar{w}}\Big)^{\zeta} \hat{b}(\bar{w}) + \Big(\frac{x}{\bar{w}^2}\Big)^{\zeta} \hat{b}(\bar{w}) + \Big(\frac{x}{\bar{w}^3}\Big)^{\zeta} \hat{b}(\bar{w}) + \hat{b}\big(\frac{x}{\bar{w}^3}\big) \\
=& x^{\zeta} \hat{b}(\bar{w}) \sum_{n=1}^{\infty} \bar{w}^{-n\zeta} + \hat{b}(0) \\
=& \frac{x^{\zeta} \hat{b}(\bar{w})}{\bar{w}^{\zeta} - 1} + \hat{b}(0) \\
=& \frac{(x^{\zeta} - 1)\hat{b}(\bar{w})}{\bar{w}^{\zeta} - 1},
\end{aligned}$$

where we've used the fact that $\hat{b}(1) = 0$. The associated solution for $b(w)$ has the form $b(w) = b_1 w^{\zeta} + b_0$. This establishes the proposition. ∎

Let $\mathcal{F}_N$ denote the restriction of $\mathcal{F}$ to allocations with N equal-measure cohorts. For an allocation $f \in \Delta_0 \cap \mathcal{F}_N$, let $\tilde{w}$ denote the random variable adapted to Σ that corresponds to the total aggregate (per capita) income allocated by f. Likewise, let $\tilde{x}_i$ denote the stochastic share-density of aggregate income allocated to cohort i made by f. Thus, f allocates the income density $\tilde{w}\tilde{x}_i$ to individual or cohort i. Note that if $g \in \Delta(f)$, then $f_{\alpha} g \in \Delta(f)$ for any α-mixture of f and g. In other words, Δ_0 is closed under α-mixtures. Because $\Delta_0 \cap \mathcal{F}_N$ is a mixture space with respect to α-mixtures, one can use Axiom 3 to apply standard results (e.g., Herstein and Milnor, 1953) and deduce the existence of a unique (up to affine transformation) utility representation. In particular, when restricted to $f \in \Delta_0 \cap \mathcal{F}_N$, this representation is pinned down by Proposition A.3 to have the form:

$$\begin{aligned} V(f) &= E\left[\sum_{i=1}^{N}\gamma_i\tilde{x}_i\tilde{w}^\zeta - \varphi\sum_{i=1}^{N}\tilde{x}_i^2\tilde{w}^\zeta + b\tilde{w}^\zeta\right] \\ &= \sum_{i=1}^{N}\gamma_i E[\tilde{x}_i\tilde{w}^\zeta] - \varphi E\Big[\Big(\sum_{i=1}^{N}\tilde{x}_i^2\Big)\tilde{w}^\zeta\Big] + bE[\tilde{w}^\zeta] \end{aligned}$$

If the exponent $\zeta \neq 1$ in the above representation, then it is possible to construct an allocation for which $E[\tilde{x}_i\tilde{w}] = E[\tilde{x}_{i+1}\tilde{w}]$ for some $i < N$, and yet $E[\tilde{x}_i\tilde{w}^\zeta] \neq E[\tilde{x}_{i+1}\tilde{w}^\zeta]$. In this case, if $\gamma_i \neq \gamma_{i+1}$ then Condition B3 would be violated upon permuting the identities of individuals i and $i+1$. Thus, if $\zeta \neq 1$ then the γ_i's must coincide. Summarizing, and recalling that $\frac{1}{N}\sum_{i=1}^{N}\tilde{x}_i \equiv \tilde{w}$, the representation must take the form:

$$V(f) = \begin{cases} \nu E[\tilde{w}^\zeta] - \varphi E\Big[\Big(\sum_{i=1}^{N}\tilde{x}_i^2\Big)\tilde{w}^\zeta\Big] & \text{if } \zeta \neq 1 \\ \sum_{i=1}^{N}\gamma_i E[\tilde{x}_i\tilde{w}] - \varphi E\Big[\Big(\sum_{i=1}^{N}\tilde{x}_i^2\Big)\tilde{w}\Big] & \text{otherwise.} \end{cases}$$

To extend the representation to $\mathcal{F} \equiv \bigcup_N \mathcal{F}_N$, first consider that, by continuity, if $f \in \mathcal{F}_N$ then it must be that $f \in \mathcal{F}_{2N}$. Denoting the coefficients corresponding to $\mathcal{F}_N$ as $\gamma_i(N)$ and $\varphi(N)$, consider an allocation $f \in \Delta_0 \cap \mathcal{F}_N$ with corresponding share density $\tilde{x}_1$ allocated to the cohort with the highest mean income. The contribution of this cohort to $V_N(f)$ is $\gamma_1(N)E[\tilde{x}_1\tilde{w}] - \varphi(N)E[\tilde{x}_1^2\tilde{w}]$. Because $f \in \mathcal{F}_{2N}$, the contribution of this cohort to V_{2N} is $\big(\gamma_1(2N) + \gamma_2(2N)\big)E[\tilde{x}_1\tilde{w}] - \varphi(2N)E[2\tilde{x}_1^2\tilde{w}]$.[21] Thus, to ensure that the representations are consistent,

$$\begin{aligned} \gamma_i(N) &= \gamma_{2i-1}(2N) + \gamma_{2i}(2N) \qquad \text{and} \\ \varphi(2N) &= \frac{\varphi(N)}{2}. \end{aligned}$$

In particular, $\varphi(N) = \frac{\varphi(1)}{N}$ while the $\gamma_i(N)$'s sum to one regardless of N. The extension to the continuum is standard. The φ term can be expressed as an integral by recalling that in state θ, $\tilde{w}\tilde{x}_i = f(\theta, p)$, where p is an element of the i^{th} cohort. The γ_i terms give rise to a rank-dependent "average", as in Yaari (1987), relying on a convex function $G : [0,1] \mapsto [0,1]$, such that $G(0) = 0$ and $G(1) = 1$. $G(\cdot)$ need not be everywhere increasing because the γ_i's are not guaranteed to be positive for every i and N. Overall, these considerations imply the following integral representation over a dense subset of $\mathcal{F}$ (i.e., letting $N \to \infty$):

$$V(f) = \begin{cases} \nu \int_\Theta w(f,\theta)^\zeta \, d\mu(\theta) - \varphi \int_{[0,1]}\int_\Theta w(f,\theta)^\zeta x(f,p,\theta)^2 \, d\mu(\theta)\, dm(p) & \text{if } \zeta \neq 1 \\ \int_0^\infty G\big(D_{\bar{f}}(w)\big)dw - \varphi \int_{[0,1]}\int_\Theta f(p,\theta)x(f,p,\theta)\, d\mu(\theta)\, dm(p) & \text{Otherwise.} \end{cases} \tag{34}$$

[21] The densities and per capita income are invariant to whether one chooses to represent f as an element of $\mathcal{F}_N$ or $\mathcal{F}_{2N}$.

Continuity then allows one to extend the representation to all of $\mathcal{F}$ and completes the proof of sufficiency of the Axioms. Necessity is trivial.

Proof of Proposition 1:

Let $f, g \in \mathcal{F}$ be non-stochastic allocations. Let $\{(\xi_i, D_i)\}$ denote the points at which the graphs of the decumulative distribution functions corresponding to f and g cross, ordered such that $\xi_1 \leq \xi_2 \ldots$, and such that $D_i < 1$.[22] It is easy to establish that Definition 2 is equivalent to requiring that

$$\int_0^{\xi_i} D_f(w)dw \geq \int_0^{\xi_i} D_g(w)dw \tag{35}$$

for each crossing point ξ_i. Now, define the convex functions,

$$G_i(z) = (1-\epsilon)\frac{\left(z - D_i\right)^+}{1 - D_i} + \epsilon z,$$

where $\epsilon > 0$ is arbitrarily small. It should be clear that $G_i(\cdot)$ satisfies the requirements in Theorem 2.

Now suppose that $f \succcurlyeq g$ according to every utility function consistent with Theorem 2 with $\varphi = 0$. Then, in particular, this is true when the $G(z) = G_i(z)$ for any i. Because ϵ is arbitrarily small, this implies that for each i,

$$\int_0^{\xi_i} \big(D_f(w) - D_i\big)dw \geq \int_0^{\xi_i} \big(D_g(w) - D_i\big)dw,$$

implying Eq. (35) and therefore that f dominates g. The fact that dominance implies $f \succcurlyeq g$ for every function of the form in Eq. (7) is established in Chew and Mao (1995).

The integral condition in Proposition 1 is required because φ can be arbitrarily large. ∎

Proof of Proposition 2: Define $V_0(\cdot)$ as $V(\cdot)$ with φ set to zero. Then $V_0(\cdot)$ depends only on the mean allocation to each cohort and one can therefore restrict the discussion to non-stochastic allocations. Here, the standard results imply that $V_0(\bar{w}) \geq V_0(f)$. In particular, setting $x(f, p, \theta) = 1$ for every $\theta \in \Theta$ and $p \in [0, 1]$ allocates $\bar{w}$ for sure to each cohort.

[22] If the graphs coincide on some closed interval, then denote only the initial point of the interval as the crossing point.

Now, write

$$\int_{[0,1]} \int_{\Theta} f(p,\theta)x(f,p,\theta)\, d\mu(\theta)\, dm(p) = \sum_{i=1}^{N}\sum_{s=1}^{M} \mu_s m_i x_{is}^2 w_s w_s,$$

Where m_i denotes the mass of cohort i and μ_s denotes the probability that cohorts 1 through N will receive the distinct set of payoffs $x_{1s}w_s$ through $x_{Ns}w_s$, respectively (with the restrictions $\sum_{i=1}^{N} m_i x_{is} = 1$, $\sum_{i=1}^{N} m_i = 1$ and $\sum_{s=1}^{M} \mu_s = 1$). Now, consider the program,

$$\max_{\{y_{is} \geq 0\}} \sum_{i=1}^{N}\sum_{s=1}^{M} \mu_s m_i y_{is}^2 w_s, \quad \text{s.t.} \quad \sum_{i=1}^{N} m_i y_{is} = 1.$$

It is easy to verify that the solution is $x_{is} = 1$. Because the objective function is clearly strictly concave, the solution is unique.

Thus $V(f)$ is dominated by the allocations $V(\bar{w})$. Moreover, uniqueness of the solution to the program above implies that the dominance is strict unless $x(f,p,\theta) = 1$ for almost every $(p,\theta) \in [0,1] \times \Theta$. ∎

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